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ANNALS OF MATHEMATICS.

VOL. I. MARCH, 1884. NO. 1.

THE DETERMINATION OF THE MASS OF A PLANET FROM THE RELATIVE POSITION OF TWO SATELLITES.

BY PROF. ASAPH HALL, Washington, D. C.

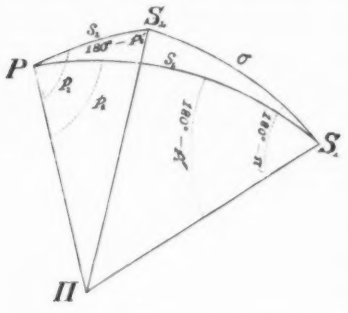
ONE of the easiest methods of determining the mass of a planet is from the observations of its satellites. The two principal elements that are needed are the periodic time of the satellite about its primary, and the semi-major axis of its orbit. Thus, if M and m are the masses of the sun and the planet, and τ_0 and τ are the periodic times of the planet and the satellite, we shall have very nearly,

$$\frac{m}{M} = \frac{\tau^2}{\tau_0^2} \cdot \frac{a^3}{a_0^3},$$

a_0 and a being the mean distances of the planet from the sun and of the satellite from its primary. The above is the simplest form of this equation, and should it be necessary in any case to consider the elliptical figure of the planet, and its mass when added to that of the sun, these small corrections can be applied easily. Of the two principal elements the periodic time of a satellite can be found with sufficient accuracy by comparing observations of it made at distant epochs, and hence the error arising from this source may be considered as vanishing. But with respect to the semi-major axis the case is different. For the determination of this element we have to depend on micrometrical measurements, and several sources of error present themselves. Thus the value in arc of one turn of the screw of the micrometer must be known very exactly, since this value will frequently be multiplied by a large factor, and of course an error in the value of the screw will also be multiplied by the same factor. With care on the part of the astronomer this error may be reduced to a very small quantity; and yet it is surprising how often such errors remain in micrometers that have been a long time in use. But there is another error in micrometrical measurements that is not easily got rid of, and this is the personal error of the observer. This error

will sometimes amount to half a second of arc, and since it is not eliminated by the repetition of observations, each observer will have his own value of the mean distance of a satellite, and hence his own mass of a planet. This personal error may be still greater, if, as is generally the case, we have to do with the disk of a planet. In this case we must bisect the disk with one of the wires, or else measure from the limb of the planet, and both operations are difficult and tend to increase the constant error of the measurement.

On account of this difficulty Struve, the present Director of the Pulkowa Observatory, has proposed that observations be made of the relative positions of two satellites for the purpose of correcting their elements, and in this way deducing the mass of the planet. Mr. Backlund, one of the astronomers of the Pulkowa Observatory, has already begun a series of measurements of the relative positions of the satellites of Jupiter with a heliometer, and it is said that these observations are two or three times as accurate as those made by reference to the limb of the planet, and moreover that the complete observation of the relative position of two satellites requires only a third part of the time that is necessary to determine the position of one of them with respect to the center of the planet. Mr. Backlund has published an outline of this method, which is theoretically very simple. Let P be the place of the planet projected on the heavens, Π the pole of the heavens, S_1 and S_2 the satellites, and denote the angles and sides of the spherical triangle, as in the figure.



The angles of position and distances are designated generally by p and s, with proper subscripts, and the relative angle of position and the distance of the two satellites by Π and σ. Then

$$\text{angle } S_1PS_2 = p_2 - p_1, \qquad \text{angle } S_2S_1P = p'_1 - \pi',$$

and the spherical triangle S_1PS_2 gives the equations,

$$\sin\sigma \sin(p'_1 - \pi') = \sin s_2 \sin(p_2 - p_1)$$
$$\sin\sigma \cos(p'_1 - \pi') = \sin s_1 \cos s_2 - \cos s_1 \sin s_2 \cos(p_2 - p_1).$$

These are exact formulas, but as s_1 and s_2 are always less than $12'$ in any real case that we have, we may generally put

$$p'_1 = p_1, \text{ and } \pi' = \pi;$$
$$\cos s_1 = \cos s_2 = 1, \text{ and } \sin s_1 = s_1, \ \sin s_2 = s_2, \ \sin\sigma = \sigma,$$

and hence we get

$$\sigma \sin(p_1 - \pi) = s_2 \sin(p_2 - p_1)$$
$$\sigma \cos(p_1 - \pi) = s_1 - s_2 \cos(p_2 - p_1),$$

and from these we have

$$\sigma \sin \pi = s_1 \sin p_1 - s_2 \sin p_2$$
$$\sigma \cos \pi = s_1 \cos p_1 - s_2 \cos p_2.$$

Now $s_1 \sin p_1$ and $s_1 \cos p_1$ are the rectangular co-ordinates of the first satellite with respect to the center of the planet, and the preceding equations become

$$x = x_1 - x_2$$
$$y = y_1 - y_2,$$

where x and y are the relative co-ordinates of the satellites; a result that might have been anticipated, since the above assumptions reduce the work to that of plane co-ordinates. Should it be necessary in any case we may use the rigorous formulas; or, what is easier, we may expand them and retain terms of the lower orders. In the present case we have therefore

$$dx = dx_1 - dx_2$$
$$dy = dy_1 - dy_2.$$

The differentials on the right hand side can be expressed in terms of the differtials of the elements of the orbits of the two satellites, and of the differential coefficients with respect to their elements; and since the left hand side will be known by comparing the computed relative position with the observed, we have the means of forming the equations of condition for correcting the elements of these orbits, and therefore for determining the mass of the planet. Thus dx_1 and dx_2 will be of the form

$$dx_1 = \frac{dx_1}{da_1} \cdot \delta a_1 + \frac{dx_1}{d\varepsilon_1} \cdot \delta \varepsilon_1 + \frac{dx_1}{dP_1} \cdot \delta P_1 + \frac{dx_1}{de_1} \cdot \delta e_1 + \frac{dx_1}{dN_1} \cdot \delta N_1 + \frac{dx_1}{dJ_1} \cdot \delta J_1$$

$$dx_2 = \frac{dx_2}{da_2} \cdot \delta a_2 + \frac{dx_2}{d\varepsilon_2} \cdot \delta \varepsilon_2 + \frac{dx_2}{dP_2} \cdot \delta P_2 + \frac{dx_2}{de_2} \cdot \delta e_2 + \frac{dx_2}{dN_2} \cdot \delta N_2 + \frac{dx_2}{dJ_2} \cdot \delta J_2$$

where a_1, ε_1, P_1, e_1, N_1, J_1 are the elements of the orbit of the first satellite, and a_2, ε_2, P_2, e_2, N_2, J_2 are those of the second satellite. The equations of condition therefore that result from this method will each contain twelve unknown quantities; and it will be necessary to arrange the observations so that there may be no indetermination in the solution. For this purpose the satellites should be observed when they are on different sides of the planet as well as when they are on the same side, and in a variety of positions.

In the common method of referring a satellite directly to the center of the planet we have in the equations of condition six unknown quantities. Now the labor of solving a set of equations of condition increases rapidly with the number of unknown quantities. If i be this number, then the number of coefficients in the normal equations is found by the summation of an arithmetical series to be

$$\frac{i\,(i+3)}{2}.$$

The number of auxiliary quantities that have to be formed in the successive elimination of the unknown quantities from the normals is given in the following table, with their first and second differences.

No.	$\triangle_1$	$\triangle_2$
$\frac{i^2+i-2}{2}$,		
	$-i+0$	
$\frac{i^2-i-2}{2}$,		$+1$
	$-i+1$	
$\frac{i^2-3i+0}{2}$,		$+1$
	$-i+2$	
$\frac{i^2-5i+4}{2}$,		$+1$
	$-i+3$	
$\frac{i^2-7i+10}{2}$,		$+1$
	$-i+4$	
$\frac{i^2-9i+18}{2}$,		
etc.		

The law of this series is evident. The sum of $i-1$ terms of the series is

$$\text{Sum} = \frac{i\,(i-1)\,(i+2)}{2} - \frac{i^2\,(i-1)}{2} + \frac{i\,(i-1)\,(i-2)}{2.3}$$
$$= \frac{(i-1)\,i\,(i+4)}{2.3}$$

Hence the whole number of quantities to be formed in this solution is

$$\frac{(i-1)\,i\,(i+4)}{2.3} + \frac{i\,(i+3)}{2} = \frac{i\,(i+1)\,(i+5)}{2.3},$$

In the case of $i=6$, we shall have 77 quantities to form, and when $i=12$, we shall have 442 quantities to form. The labor of solution is nearly six times as great in the new method, but this is a consideration that is not perhaps entitled to much weight. Again, when the number of unknown quantities is increased, the probable error of a single equation is generally increased, but this increase will be small when the number of observations is large; and if the observations be well arranged, this question may be left out of consideration. On the other hand, the gain in accuracy seems to be substantial, and the new method promises a real advance, especially in the case of the satellites of Jupiter and Saturn. Another advantage is that it is a different method of making the observations, and in this way it offers an independent check on the old methods, and may be the means of eliminating some constant errors.

ON THE LUNAR INEQUALITIES PRODUCED BY THE MOTION OF THE ECLIPTIC.

By Dr. G. W. Hill, Washington, D. C.

This subject has been treated by Hansen,* and more recently by Sir G. B. Airy and Prof. J. C. Adams.† Hansen's discussion is accommodated to the peculiar system of co-ordinates he employs, and the two later writers do not consider the inequalities in longitude. Hence an investigation, giving the inequalities of the latitude and longitude, at first, in the literal form, may be of value. The procedures employed are very similar to those of Pontécoulant, and are doubtless not as direct as might be imagined. The paper was written as long ago as 1867.

I.

Expressed in the ordinary notation, when the co-ordinates are referred to fixed planes, the differential equations of motion are

$$\frac{d^2X}{dt^2}+\frac{\mu}{r^3}X=\frac{dR}{dx},$$
$$\frac{d^2Y}{dt^2}+\frac{\mu}{r^3}Y=\frac{dR}{dy},$$
$$\frac{d^2Z}{dt^2}+\frac{\mu}{r^3}Z=\frac{dR}{dz}.$$

Since the directions of the axes are arbitrary, let the axis of X be directed toward the ascending node of the moving ecliptic on the ecliptic of 1850; and let the axis of Z be perpendicular to the latter plane. Taking now another system of co-ordinates, x, y, and z, such that the axis of x has the same direction as that of X, but the axis of z is perpendicular to the moving ecliptic, let $\pi\,(t-1850)$ be the inclination of the moving ecliptic to that of 1850; then, neglecting quantities of the order of π^2, these equations exist,

$$X=x,$$
$$Y=y-\pi\,(t-1850)\,z,$$
$$Z=z+\pi\,(t-1850)\,y.$$

The differential equations of motion, expressed in terms of the second system of co-ordinates, are

$$\frac{d^2x}{dt^2}+\frac{\mu}{r^3}x=\frac{dR}{dx},$$
$$\frac{d^2y}{dt^2}+\frac{\mu}{r^3}y=\frac{dR}{dy}+2\pi\frac{dz}{dt},$$
$$\frac{d^2z}{dt^2}+\frac{\mu}{r^3}z=\frac{dR}{dz}-2\pi\frac{dy}{dt}.$$

* *Darlegung* etc., Arts. 175–178.

† *Monthly Notices*, Vol. XLI., pp. 264, 375, and 385.

Denoting the true longitude of the moon by λ, from these may be derived the two

$$\frac{d^2r^2}{2\,dt^2} - \frac{\mu}{r} + \frac{\mu}{a} = 2\int d'R + r\frac{dR}{dr} + 2\pi\frac{ydz - zdy}{dt},$$

$$\frac{d.(r^2 - z^2)\frac{d\lambda}{dt}}{dt} = \frac{dR}{d\lambda} + 2\pi\frac{xdz}{dt}.$$

In this discussion all terms involving the solar eccentricity and parallax will be neglected. Let ζ denote the moon's mean angular distance from a point 90° behind the ascending node of the moving ecliptic on the ecliptic of 1850, or $\zeta = \varepsilon + nt - \Pi + 90°$. For simplicity, the semi-axis major of the lunar orbit will be made equal to unity, and, as usual in the lunar theory, m will be written for $\frac{n'}{n}$. Also let φ and τ denote respectively the true and mean angular distance of the moon from the sun.

With these restrictions and notation

$$2\int d'R + r\frac{dR}{dr} = 4R + 2m\int\frac{dR}{d\lambda}d\zeta,$$

$$R = \frac{m^2}{4}[3(r^2 - z^2)\cos 2\varphi + r^2 - 3z^2],$$

$$\frac{dR}{d\lambda} = -\frac{3}{2}m^2(r^2 - z^2)\sin 2\varphi.$$

If the symbol δ prefixed to any quantity denote that part of it, in its development in series, which is multiplied by the first power of π, the equations for determining δr, $\delta\lambda$, and δz are

$$\frac{d^2r\delta r}{d\zeta^2} + \frac{\mu r\delta r}{n^2r^3} = 4\delta R + 2m\int\delta.\frac{dR}{d\lambda}d\zeta + 2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta},$$

$$\frac{r^2d.\delta\lambda}{d\zeta} + 2\frac{d\lambda}{d\zeta}(r\delta r - z\delta z) = \int\delta.\frac{dR}{d\lambda}d\zeta + 2\frac{\pi}{n}\int xdz,$$

$$\frac{d^2\delta z}{d\zeta^2} + \left(\frac{\mu}{n^2r^3} + m^2\right)\delta z = -2\frac{\pi}{n}\frac{dy}{d\zeta}.$$

In these equations terms multiplied by the square and higher powers of the inclination of the moon's orbit are neglected; and, since δr and $\delta\lambda$ are multiplied by the first power of this quantity, this involves the neglect of terms such as $z\delta r$ and $z\delta\lambda$. For the same reason all higher powers of z than the second have been omitted in R.

These equations suffice to determine the inequalities we seek; but for a term of long period in $\delta\lambda$ it will be more commodious to employ another equation. We have

$$r\frac{d^2r}{d\zeta^2} - (r^2 - z^2)\frac{d\lambda^2}{d\zeta^2} - \left(\frac{dz}{d\zeta} - \frac{z}{r}\frac{dr}{d\zeta}\right)^2 + \frac{\mu}{n^2r} = r\frac{dR}{dr} + 2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta},$$

or

$$\frac{r^2-z^2}{r^2}\frac{d\lambda^2}{d\zeta^2}-\frac{1}{r}\frac{d^2r}{d\zeta^2}+\left(\frac{1}{r}\frac{dz}{d\zeta}-\frac{z}{r^2}\frac{dr}{d\zeta}\right)^2-\frac{\mu}{n^2r^3}=-2\frac{R}{r^2}-2\frac{\pi}{n}\frac{ydz-zdy}{r^2d\zeta}.$$

Taking the variation with respect to π, and then multiplying by r^2,

$$\left\{\begin{array}{l} 2r^2\dfrac{d\lambda}{d\zeta}\dfrac{d.\delta\lambda}{d\zeta}-2\dfrac{d\lambda^2}{d\zeta^2}z\delta z-\dfrac{rd^2\delta r-d^2r.\delta r}{d\zeta^2} \\ +2\left(\dfrac{dz}{d\zeta}-\dfrac{z}{r}\dfrac{dr}{d\zeta}\right)\left(\dfrac{d.\delta z}{d\zeta}-\dfrac{dr\delta z}{rd\zeta}\right)+\dfrac{3\mu r\delta r}{n^2r^3} \end{array}\right\}=\left\{\begin{array}{l} -2\dfrac{dR}{d\lambda}\delta\lambda-2\dfrac{dR}{dz}\delta z \\ -2\dfrac{\pi}{n}\dfrac{ydz-zdy}{d\zeta} \end{array}\right\}$$

But

$$3\frac{d^2r\delta r}{d\zeta^2}+3\frac{\mu r\delta r}{n^2r^3}=12\delta R+6m\int\delta.\frac{dR}{d\lambda}d\zeta+6\frac{\pi}{n}\frac{ydz-zdy}{d\zeta};$$

subtracting this

$$r^2\frac{d\lambda}{d\zeta}\frac{d.\delta\lambda}{d\zeta}=\frac{d\left[2d(r\delta r)-dr.\delta r\right]}{d\zeta^2}+\frac{d\lambda^2}{d\zeta^2}z\delta z-\frac{dz}{d\zeta}\frac{d.\delta z}{d\zeta}+\frac{d(z\delta z)}{d\zeta}\frac{dr}{rd\zeta}$$
$$-\left(\frac{dr}{rd\zeta}\right)^2z\delta z-7\delta R+\frac{dR}{dr}\delta r-3m\int\delta.\frac{dR}{d\lambda}d\zeta-4\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}.$$

In determining δz we shall stop at terms of the order $m\frac{\pi}{n}$, and shall neglect all terms multiplied by powers of the lunar eccentricity e higher than the first. In $\delta\lambda$ we shall neglect e altogether; and, since the inequalities in the lunar parallax resulting from δr are insensible, δr will be determined only so far as it is necessary to the determination of $\delta\lambda$. Let ξ denote the moon's mean anomaly, and η its mean argument of latitude, or $\eta=\varepsilon+nt-\Omega$. In applying the last equation to determining the co-efficient of $\sin(\zeta-\eta)$ in $\delta\lambda$ to terms of the order of $\gamma\frac{\pi}{n}$, (where γ denotes the same function of the inclination as it does in Pontécoulant's *Théorie Analytique*), it will be necessary to compute each member to terms of the order of $m^2\gamma\frac{\pi}{n}$. But $r\delta r$ is of the order of $\gamma\frac{\pi}{n}$, consequently $\frac{d^2r\delta r}{d\zeta^2}$, in the term which has $\zeta-\eta$ for its argument, is of the order of $m^4\gamma\frac{\pi}{n}$, and thus may be neglected; moreover $\frac{dr}{d\zeta}$ is of the order of m^2, and hence $\frac{d(dr.\delta r)}{d\zeta^2}$ is of the order of $m^4\gamma\frac{\pi}{n}$ in the term having the same argument; this may then be also omitted.

With these simplifications the last equation becomes

$$r^2\frac{d\lambda}{d\zeta}\frac{d.\delta\lambda}{d\zeta}=\frac{d\lambda^2}{d\zeta^2}z\delta z-\frac{dz}{d\zeta}\frac{d.\delta z}{d\zeta}+\frac{dr}{rd\zeta}\frac{d(z\delta z)}{d\zeta}-\left(\frac{dr}{rd\zeta}\right)^2z\delta z$$
$$+\frac{21}{2}m^2(1+\cos 2\varphi)z\delta z-4\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}$$
$$-3m^2(1+3\cos 2\varphi)r\delta r-7\frac{dR}{d\lambda}\delta\lambda-3m\int\delta.\frac{dR}{d\lambda}d\zeta.$$

If, for brevity, we write

$$A = \frac{\mu}{n^2 r^3} + m^2,$$

$$B = \frac{\mu}{n^2 r^3} - 2m^2 - 6m^2 \cos 2\varphi,$$

$$C = 6m^2 r^2 \sin 2\varphi,$$

$$D = 3m^2 r^2 \cos 2\varphi,$$

$$E = 3m^2 \sin 2\varphi,$$

$$U = -2\frac{\pi}{n}\frac{dy}{d\zeta},$$

$$U' = 2\frac{\pi}{n}\frac{ydz - zdy}{d\zeta} - 6m^2(1 + \cos 2\varphi)\, z\delta z + 6m^3 \int \sin 2\varphi .\, z\delta z\, d\zeta,$$

$$U'' = 2\frac{\pi}{n}\int x dz + 2\frac{d\lambda}{d\zeta} z\delta z + 3m^2 \int \sin 2\varphi . z\delta z d\zeta,$$

the term $2m\int Er\delta r d\zeta$ in the equation for $r\delta r$ being omitted as not giving any terms which we wish to preserve, and it being sufficient to put $B = 1$, and $\frac{d\lambda}{d\zeta} = 1$, where the latter multiplies $r\delta r$ in the equation for $\delta\lambda$, the three equations become

$$\frac{d^2\delta z}{d\zeta^2} + A\delta z = U,$$

$$\frac{d^2 r\delta r}{d\zeta^2} + r\delta r + C\delta\lambda + 2m\int D\delta\lambda d\zeta = U',$$

$$r^2\frac{d.\delta\lambda}{d\zeta} + 2\, r\delta r + \int [D\delta\lambda + Er\delta r]\, d\zeta = U''.$$

To the degree of approximation we desire

$$\frac{1}{r} = 1 + \frac{1}{6}m^2 + (m^2 + \frac{19}{6}m^3)\cos 2\tau,$$

$$\lambda = \varepsilon + nt + \left(\frac{11}{8}m^2 + \frac{59}{12}m^3\right)\sin 2\tau.$$

Also (*Pontécoulant, Théorie Analytique*, Tom. IV., pp. 216, 226)

$$A = 1 + \frac{3}{2}m^2 - \frac{9}{32}m^4 + \frac{55}{16}m^5 + (3m^2 + \frac{19}{2}m^3 + \frac{137}{6}m^4)\cos 2\tau$$
$$+ (3 + \frac{3}{2}m^2)\, e\cos\xi + \left(\frac{45}{8}m + \frac{657}{32}m^3\right) e\cos(2\tau - \xi) + \frac{147}{16}m^2 e\cos(2\tau + \xi).$$

From $y = r \sin(\lambda - \Pi)$ we derive

$$y = -\left(1 - \frac{m^2}{6}\right) \cos\zeta + \frac{19}{16} m^2 \cos(\zeta - 2\tau) - \frac{1}{2} e \cos(\zeta + \tilde{\varpi}),$$

$$U = -\left(2 - \frac{m^2}{3}\right) \frac{\pi}{n} \sin\zeta - \frac{19}{8} m^2 \frac{\pi}{n} \sin(\zeta - 2\tau) - 2e \frac{\pi}{n} \sin(\zeta + \tilde{\varpi}).$$

Let

$$\delta z = \frac{\pi}{n} \Big\{ A_1 \sin\zeta + A_2 \sin(\zeta - 2\tau) + A_3 \sin(\zeta + 2\tau) + A_4 \sin(\zeta - 4\tau)$$
$$+ A_5 e \sin(\zeta - \tilde{\varpi}) + A_6 e \sin(\zeta + \tilde{\varpi}) + A_7 e \sin(\zeta - 2\tau + \tilde{\varpi}) + A_8 e \cos(\zeta + 2\tau - \tilde{\varpi})$$
$$+ A_9 e \sin(\zeta - 2\tau - \tilde{\varpi}) + A_{10} e \sin(\zeta + 2\tau + \tilde{\varpi}) + A_{11} e \sin(\zeta - 4\tau + \tilde{\varpi}) \Big\}.$$

On substituting this expression in the first of the three differential equations, the following equations result for determining A_1, A_2, etc.,

$$\left(\frac{3}{2}m^2 - \frac{9}{32}m^4 + \frac{55}{16}m^5\right) A_1 + \left(\frac{3}{2}m^2 + \frac{19}{4}m^3 + \frac{137}{12}m^4\right)(A_2 + A_3) = -2 + \frac{1}{3}m^2,$$

$$\left(4m - \frac{5}{2}m^2\right) A_2 + \left(\frac{3}{2}m^2 + \frac{19}{4}m^3 + \frac{137}{12}m^4\right) A_1 = -\frac{19}{8}m^2,$$

$$-(8 - 12m) A_3 + \left(\frac{3}{2}m^2 + \frac{19}{4}m^3\right) A_1 = 0,$$

$$-8A_4 + \frac{3}{2}m^2 A_2 = 0,$$

$$\left(1 + \frac{3}{2}m^2\right) A_5 + \left(\frac{3}{2} + \frac{3}{4}m\right) A_1 + \frac{45}{16} m A_2 = 0,$$

$$-\left(3 - \frac{9}{2}m^2\right) A_6 + \left(\frac{3}{2} + \frac{3}{4}m\right) A_1 = -2,$$

$$A_7 + \frac{3}{2}m^2 A_6 + \frac{3}{2}A_2 + \left(\frac{45}{16}m + \frac{657}{64}m^2\right) A_1 = 0,$$

$$-(3 - 8m) A_8 + \frac{3}{2}m^2 A_5 + \frac{3}{2}A_3 + \left(\frac{45}{16}m + \frac{657}{64}m^2\right) A_1 = 0,$$

$$-(3 - 8m) A_9 + \frac{3}{2}m^2 A_5 + \frac{3}{2}A_2 + \frac{147}{32}m^2 A_1 = 0,$$

$$-15 A_{10} + \frac{3}{2}m^2 A_6 + \frac{3}{2}A_3 + \frac{147}{32}m^2 A_1 = 0,$$

$$-3 A_{11} + \frac{45}{16} m A_2 = 0.$$

By the solution of these, this expression of δz is obtained.

$$\delta z = -\left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{31}{9} + \frac{3395}{288}m\right) \frac{\pi}{n} \sin\zeta$$
$$+ \left(\frac{1}{2}m^{-1} + \frac{25}{12} + \frac{1843}{288}m\right) \frac{\pi}{n} \sin(\zeta - 2\tau)$$

$$
\begin{aligned}
&- \left(\frac{1}{4} + \frac{121}{96} m\right) \frac{\pi}{n} \sin(\zeta + 2\tau) \\
&+ \frac{3}{32} m \frac{\pi}{n} \sin(\zeta - 4\tau) \\
&+ \left(2 m^{-2} + \frac{3}{4} m^{-1} + \frac{169}{96}\right) e \frac{\pi}{n} \sin(\zeta - \xi) \\
&- \left(\frac{2}{3} m^{-2} + \frac{1}{4} m^{-1} + \frac{43}{18}\right) e \frac{\pi}{n} \sin(\zeta + \xi) \\
&+ \left(3 m^{-1} + \frac{415}{32}\right) e \frac{\pi}{n} \sin(\zeta - 2\tau + \xi) \\
&- \left(\frac{5}{4} m^{-1} + \frac{719}{96}\right) e \frac{\pi}{n} \sin(\zeta + 2\tau - \xi) \\
&+ \left(\frac{1}{4} m^{-1} - \frac{9}{24}\right) e \frac{\pi}{n} \sin(\zeta - 2\tau - \xi) \\
&- \frac{1}{2} e \frac{\pi}{n} \sin(\zeta + 2\tau + \xi) \\
&+ \frac{15}{32} e \frac{\pi}{n} \sin(\zeta - 4\tau + \xi).
\end{aligned}
$$

The value of z (*Théorie Analytique*, Tom. IV., pp. 237, 244) is

$$
\begin{aligned}
z = \gamma \Big\{ &\left(1 - \frac{m^2}{6} + \frac{57}{128} m^3\right) \sin \eta + \left(\frac{3}{8} m + \frac{41}{32} m^2 + \frac{5293}{1536} m^3\right) \sin(2\tau - \eta) \\
&+ \left(\frac{3}{16} m^2 + \frac{7}{8} m^3\right) \sin(2\tau + \eta) \Big\},
\end{aligned}
$$

whence, by multiplication, is obtained

$$
\begin{aligned}
z\delta z = &- \left(\frac{2}{3} m^{-2} + \frac{1}{4} m^{-1} + \frac{491}{288}\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta) \\
&+ \left(\frac{1}{4} m^{-1} + \frac{11}{12} + \frac{91}{36} m\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta - 2\tau) \\
&+ \left(\frac{1}{4} m^{-1} + \frac{79}{76} + \frac{6067}{2304} m\right) \gamma \frac{\pi}{n} \cos(\zeta - \eta + 2\tau) \\
&+ \left(\frac{2}{3} m^{-2} + \frac{1}{4} m^{-1} + \frac{29}{18}\right) \gamma \frac{\pi}{n} \cos(\zeta + \eta) \\
&- \left(\frac{1}{2} m^{-1} + \frac{191}{96} + \frac{14795}{2304}\right) \gamma \frac{\pi}{n} \cos(\zeta + \eta - 2\tau) \\
&+ \frac{1}{4} \gamma \frac{\pi}{n} \cos(\zeta + \eta + 2\tau) \\
&+ \frac{3}{32} \gamma \frac{\pi}{n} \cos(\zeta + \eta - 4\tau).
\end{aligned}
$$

[TO BE CONTINUED.]

RECTILINEAR ASYMPTOTES.

By Prof. E. B. Smith, Richmond, Va.

1. The radius-vector from the origin to a point at infinity is parallel to the asymptote containing that point. Hence, if from the equation to any such radius, $y = \mu x$, we substitute for y in the algebraic equation to a curve, $\varphi(x, y) = 0$, and from $\varphi(x, \mu x) = 0$ ascertain a value μ_∞ corresponding to $x = \infty$, we obtain the inclination of the asymptote to the x-axis.

2. The intercept of the asymptote on the y-axis is equal to the difference β between the ordinate of the asymptote and that of the radius-vector $y = \mu_\infty x$, for any common abscissa. Or, as $\beta = \mu x - \mu_\infty x$,

$$\beta_\infty = \frac{\mu - \mu_\infty}{\left(\frac{1}{x}\right)} = \frac{d\mu}{d\left(\frac{1}{x}\right)} \text{ when } x = \infty,$$

since in φ, $y = \mu x$ is the ordinate of the curve which becomes the ordinate of the asymptote when the abscissa becomes infinite. Thus, the equation to the asymptote is $y = \mu_\infty x + \beta_\infty$.

3. It is obvious that no terms in φ under the highest or n^{th} degree enter into the determination of finite values of μ. If there be infinite values, that is, if there be asymptotes parallel to the y-axis, these can be found by inspection of the co-efficient of the highest powers of y in φ.

4. If φ be rational, since only finite values of β_∞ are possible for a finite asymptote not parallel to the y-axis, in the investigation of these, besides the terms of the n^{th} degree in $\varphi(x, \mu x)$ only those of the $(n-1)^{th}$ degree are involved. Let these terms be respectively $x^n f\mu$ and $x^{n-1} f_1 \mu$; then $\frac{\varphi(x, \mu x)}{x^n} = f\mu + \frac{1}{x} f_1 \mu + \frac{1}{x^n} f_2 = 0$, and μ_∞ is found from $f = 0$ and $\beta_\infty = -\frac{f_1}{\left(\frac{df}{d\mu}\right)}$

5. If the coëfficients of the terms of the two highest degrees in φ be none of them zero except that of x^n, one value of μ_∞ is 0 and a line parallel to the x-axis is an asymptote. If, at the same time, f_1 has μ for a factor, that is, if φ has no pure x term of the degree $n-1$, then $\beta_\infty = 0$, and the x-axis is itself the asymptote. The conditions for this result are then, the absence only of the pure x terms of the two highest degrees in φ. If, also, when n is not less than 4, the two lowest powers of x (namely, the first and the zero power) be wanting, the x-axis is also a tangent, and thus a tangent and an asymptote coïncide. If $n = 3$, the cubic consists of a conic and the x-axis, and the latter may be regarded as the limiting form of a coïncident tangent and asymptote.

6. If φ be a homogeneous n^{ic} the origin will be a multiple point of the order

n, and the different values of μ will each be the same for every value of x, and hence the equation is that of a system of n right lines through the origin, each of which may be considered the final form of a coïnciding tangent and asymptote.

7. The general condition for such a coïncidence may be thus stated:

If $y=\mu_\infty x+{}_i\beta_\infty$ is a tangent to $\varphi(x,y)=0$, at $x=k$, then $\varphi(x,y)=(y-\mu_\infty x-{}_i\beta_\infty)\varphi_1+(x-k)^2\varphi_2=0$, and therefore $(x-k)^2\varphi_2=0$, or the remainder on division of φ by $y-\mu_\infty x-{}_i\beta_\infty$, equated to zero, will yield the abscissa of contact as one of a pair of equal roots.

[It should be observed, with reference to article 4, that, if $f=0$ have equal roots, $\frac{df}{d\mu}$ will be 0 when $\mu=\mu_\infty$. In this case there will be no asymptote at a finite distance unless f_1 is then also 0. In that case ${}_i\beta_\infty$ will be determined from the relation ${}_i\beta_\infty=\lim\frac{-2f_2}{\frac{df_1}{d\mu}}$, $[\mu=\mu_\infty]$, and so on.

Again, with reference to article 7, it is to be noted that $y-\mu_\infty x-{}_i\beta_\infty=0$ is tangent to φ at $x=k$; but it will not be also an asymptote unless φ_2 is three degrees lower than φ_1, for the limit of the ratio $(x-k)^2\frac{\varphi_2}{\varphi_1}$, $[x=\infty]$, can not otherwise be zero.—Ed.]

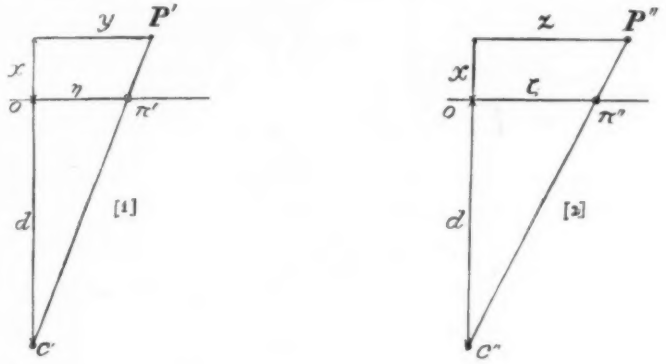
CONSTRUCTION OF PERSPECTIVE PROJECTIONS.

By Prof. Wm. M. Thornton, University of Virginia.

The common experience of draftsmen as to the laboriousness and tediousness of the ordinary processes for the construction of perspective projections has induced geometers to introduce numerous improvements into these processes. The objects aimed at in these improvements have been the reduction of the amount of purely constructive drawing, which in the ordinary processes largely exceeds that which appears in the net result or picture; the restriction of this constructive work to a narrower space, with a view to the enlargement of the picture which could be constructed on a given sheet; and the increase of accuracy in the determination of the outermost points of the figure, which owing to the obliquity of the intersections frequently left much to be desired. The improvements have been effected by using plans and elevations constructed on different sheets from that which was to contain the picture; and by employing different scales for the various sets of lines used in the construction, so as to dispense with the employment of remote vanishing points. The works of Adhemar exhibit these improved methods in perhaps their best form.

The process explained in this note is that which I habitually expound to

my students in engineering. After careful comparative tests of the various processes in the drawing-room, I have concluded that it offers the highest guarantees of accuracy and permits the most rapid and facile execution of the drawings. As will be seen, it is an attempt to effect an alliance between purely graphical methods and arithmetical processes, the draftsman being left free to use that method at each point of the work which seems to lead most directly and surely to the desired result.



I refer the points of the original to an orthogonal co-ordinate system of three planes. The x-plane is the perspective plane; the y-plane is the vertical plane at right angles thereto through the center of projection; the z-plane is the horizontal plane through the center of projection. The distances of P from these planes are x, y, z; the perspective is referred to the axes traced by the y and z-planes on the x-plane; and the distances of π, the perspective of P, from these axes are η, ζ. The distance of the point of view from the perspective plane is d. The figures show the relations of these parts [1] in plan, [2] in elevation, and furnish at once the fundamental formulæ

$$\eta = \frac{y}{1 + \frac{x}{d}}, \qquad \zeta = \frac{z}{1 + \frac{x}{d}}.$$

It is easy always so to adjust the scale that d shall be 100 or 1000.

These formulæ may be used to compute both co-ordinates of π; or to compute one co-ordinate, the point being then located by intersection on a perpendicular, a diagonal, or some line of the figure already constructed.

In particular they give for the vanishing point of a set of parallel lines whose common azimuth is α and declivity δ

$$\eta = d \tan \alpha, \qquad \zeta = d \tan \delta \sec \alpha.$$

Such points furnish invaluable tests of the accuracy of the delineation.

SOLUTION OF THE QUARTIC EQUATION,

$$x^4+Ax+B=0.$$

By Mr. A. M. Sawin, Evansville, Wis.

From Harriot's laws we easily derive the root relations

$$x=-a,\ x=-b,\ x=\tfrac{1}{2}\left[a+b\pm\sqrt{(-3a^2+2ab+3b^2)}\right],$$

whence

$$x^4+(a^3+a^2b+ab^2+b^3)x+(a^3b+a^2b^2+ab^3)=0,$$
$$\therefore\ A=a^3+a^2b+ab^2+b^3,$$
$$B=a^3b+a^2b^2+ab^3.$$

Put y for $a-b$ and z for $a+b$; we get

$$z^3+zy^2=2A,$$
$$3z^4-2z^2y^2-y^4=16B.$$

Eliminating y we get the cubic resolvent in the form

$$z^6-4Bz^2-A^2=0,$$

from which z can be found. We then find y from the relation

$$y^2=\frac{2A}{z}-z^2,$$

and calculating a and b we can form at once the four roots. The form of the equation is interesting from its occurrence in occasional problems in the resistance of materials and in hydraulic engineering.

[The root combination here used was employed first by Bette in an obscure dissertation (*Neue Auflösung der biquadratischen Gleichungen*, 1854). The form of resolvent obtained is that due to Descartes.—Ed.]

APPROXIMATE EXTRACTION OF ROOTS.

By Mr. Henry Heaton, Atlantic, Ia.

This method is based on the remark that as the n^{th} root of a number is one of its n equal factors, the average of n nearly equal factors will be an approximation to that root. For example, 2 is a first approximation to the fifth root of 40, and since

$$40=2\times2\times2\times2\times2.5,$$
$$2.1=\tfrac{1}{5}(2+2+2+2+2.5)$$

is a still closer approximation. This may be made the basis of a new approximation, and so on. More generally, if $x+h$ be the n^{th} root of R,

$$\frac{z}{z}=\frac{(n-1)x+\dfrac{R}{x^{n-1}}}{n}=x+h+(n-1)\frac{h^2}{2x},\ \text{approximately,}$$

differs from the true root by a small quantity of the second order only. Since we can always take for x the nearest whole number to the root considered, h will be a proper fraction less than $\frac{1}{2}$, and the error committed will be less than

$$\frac{n-1}{8x};$$

that is, less than $\frac{5}{4}(n-1)$ units in the m^{th} place of decimals, m being the number of integral places in x.

The principal value of the method will be found in extending the limits of application of ordinary tables of roots. For example, to find

$$\sqrt[3]{3.141593}$$

we take from a table of cubes 1.46 as the first approximation. We find

$$3.141593 \div 1.46^2 = 1.47382;$$
$$\therefore \xi = 1.46461$$

within $2\frac{1}{2}$ units of the last place. The true root is 1.46459. The next approximation would give nine places, and so on.

MR. JAMES GLAISHER'S FACTOR TABLES AND THE DISTRIBUTION OF PRIMES.

By Prof. W. W. Johnson, Annapolis, Md.

The publication of Mr. James Glaisher's *Factor Table for the Sixth Million** marks the completion of a work of extraordinary extent, and of the highest value in the theory of numbers, to which France, Germany, and England have contributed equal portions of the tireless zeal of the skillful calculator; namely, the tables which enable us to resolve into its prime factors every number less than 9,000,000. The present volume is noteworthy also as containing, in the form of an introduction of more than 100 quarto pages, a very able memoir upon the distribution of primes, embodying in tabular form the results of profound and laborious research.

The history of the factor tables is briefly as follows: The subject of the construction of extensive factor tables was agitated to a considerable extent during the latter part of the last century. Euler published a memoir in 1774 on the mode of construction of a factor table to extend to a million, and four years earlier, Lambert, in a work containing a description of a method of forming such a table, gave a folding sheet on which were given all the simple factors

* *Factor Table for the Sixth Million, containing the least factor of every number not divisible by* 2, 3 *or* 5 *between* 5,000,000 *and* 6,000,000. By James Glaisher, F. R. S. London: *Taylor and Francis*, 1883.

of numbers not divisible by 2, 3 or 5 from 1 to 10,200, and he urgently appeals to calculators in the hope that some one will extend this table to 102,000, or "if he will make himself an immortal name," to 1,020,000. But it was not until 1810 that a complete factor table of the first million appeared. This was the *Cribrum Arithmeticum* of Chernac, which gives all the prime factors of every number not divisible by 2, 3 or 5 up to 1,020,000. Chernac speaks of this as the labor, not of months, but of years. Three years later, Burckhardt published his table for the second million, or more strictly from the limit of Chernac to 2,028,000. This gives the least factor only, which obviously effects an enormous saving in bulk, while impairing the usefulness of the table only to a slight extent, the process being of course to divide by the tabular factor and enter the table for the first million with the quotient. To facilitate the divisions, a table of the first nine multiples of all primes up to 1423 was given on the first page. In 1816 Burckhardt published the continuation of this table to 3,036,000, and in 1817 he published the first million in a table uniform with his other volumes. He states that this table is based upon a comparison of a manuscript by Schenmark, in the possession of the library of the Institute, with Chernac, the latter being found correct in nearly every case of discrepancy. Burckhardt expresses his willingness to prepare the manuscript of the fourth, fifth, and sixth millions, if the sale of the three millions, which were also published in a single volume, shall appear to the publisher sufficiently favorable.

In 1850 Gauss wrote to Zacharias Dase, who was possessed of a great natural talent for computation, to repeat, as he says, in writing what he had before told him by word of mouth. He refers in this letter to a manuscript of the fourth, fifth and sixth millions which had been presented by Crelle to the Berlin Academy, and adds that he doubts not that this will sooner or later be published; he therefore urges Dase to undertake the calculation of the next four millions. In 1860, through the support of the patrons of science in his native town, Hamburg, Dase was enabled to devote himself entirely to carrying out Gauss's project, but in 1861 he died suddenly, leaving the seventh million complete and the eighth nearly complete, as well as a great part of the work for the ninth and tenth millions. The seventh million volume, in the preface of which Gauss's letter is printed, was published in 1862 by the Committee of the Dase-Stiftung. Dr. Rosenberg, of Hamburg, undertook the continuation of the work, and the eighth million was published in 1863. The ninth million, described on the title page as by Zacharias Dase and completed by Dr. H. Rosenberg, followed in 1865; a short preface signed "Das Comite der Dase-Stiftung" states that the tenth million is near completion. This million has, however, never been published.

In the report, for 1878, of the Committee on Mathematical Tables to the

British Association for the Advancement of Science, it is stated that, the committee deeming it very desirable that the gap of three millions between Burckhardt's and Dase's tables should be filled up, Prof. Cayley, the chairman of the committee, communicated early in 1877 with the Berlin Academy, with regard to the probability of the publication of the manuscript of Crelle, and learned that it had been examined and found to be so inaccurate that "the Academy was convinced that its publication would never be advisable." The calculation was then at once commenced by Mr. James Glaisher, a member of the committee, with the assistance of two computers. The table for the fourth million was published in 1879, with an extensive introduction largely historical, from which the above summary is chiefly taken. The fifth million was published in 1880, and the printing of the sixth million was commenced in that year, but the elaborate character of the introduction has delayed the appearance of this volume until 1883.

It has been learned that the manuscript of the tenth million, commenced by Dase and completed by Rosenberg, was presented in 1878 to the Berlin Academy by the widow of Dr. Rosenberg.

The arrangement of the tables is uniform for the nine millions. Owing to the rejection of the multiples of 2, 3 and 5, the last two figures of the numbers retained recur in a cycle of 300, there being 80 numbers in 300 retained. The quarto page admitting of 80 lines, the terminal figures of the argument are printed at the side of the page in three groups, each corresponding to 100 numbers. The columns are thus divided each into three sections, each section being headed by the two figures of the argument preceding those found at the side of the page. The entries in the columns are the least factors in the case of composite numbers and bars in the case of prime numbers. The mode of construction is due to Burckhardt, but being very briefly described by him is explained in some detail by Mr. Glaisher in the introduction to the fourth million. It is evident that, were all the numbers retained in the table, the multiples of any prime p would occur in certain diagonal straight lines, and therefore could be determined mechanically. The rejection of the multiples of 2, 3 and 5 dislocates this arrangement, but Burckhardt perceived that the arrangement of the multiples in the first p columns would still be repeated in the next p columns, and so on in each successive block of p columns. He therefore prepared his manuscript on sheets having the terminal figures printed at the side, and containing 77 columns, a multiple of 7 and 11 being chosen, so that the entries of the factors 7 and 11 would recur in the same position on each sheet. Thus the factors 7 and 11 could be printed upon the sheets, there being 880 entries of 7, and 480 entries of 11 on each sheet. This not only saves a great amount of labor, but ensures the accuracy of these 1360 entries on each sheet.

In like manner the multiples of 13 recur in the same positions in successive blocks of 13 columns. If a piece of paper of the size of 13 columns of the sheet, having the spaces which correspond to the multiples of 13 in the first 13 columns cut out, be laid upon these columns, the entry 13 may be made in all of the spaces of the sheet which appear through the holes unless they already contain a 7 or an 11. The screen thus constructed, which is called a "sieve," is then placed upon the next block of 13 columns and the entry 13 made through the holes, and so on. A sieve for 17 is next used in a similar manner, and the entry 17 made in those spaces which appear through the holes and are not already occupied by 7, 11 or 13. Sieves were thus constructed for all prime numbers up to 307. A little consideration will show that each sieve will have 80 holes, one in each of the 80 lines, and this was a useful check in their construction. In practice Mr. Glaisher found it more convenient to employ for the small numbers sheets consisting of several sieves in juxtaposition (thus the 13,s were entered by a sieve of 78 columns, containing 6 copies of the sieve), and for the large primes the sieves were constructed from several sheets which were not fastened together, but numbered and used successively, each sheet of the sieve, of course, in general overlapping two of the sheets of the table.

For primes greater than 307 another method was employed. It is obvious that every number for which a prime greater than 307 is the *least* factor, and within the limits of the table, is the product of two primes. To determine these multiples, in the case for example of 311, we have therefore to multiply 311 by successive prime numbers. The first multiple occurring in the fourth million was $9649 \times 311 = 3{,}000{,}839$; the series of primes commencing at 9649, is 9649, 9661, 9677, . . . , the differences being 12, 16, 2, . . . ; adding to 3,000,839 successively 12×311, 16×311, 2×311, . . . we have the multiples required. The multiples of the larger primes were thus prepared in books, from which the entries upon the sheets were made in the spaces indicated, after the sieves had been used. These spaces should, of course, all be found vacant, and this afforded a verification of the correct placing of the sieves.

The introduction to the fourth million contains a list of primes up to 30,341, with their differences.

Considered solely with reference to the frequency with which the results may be required, it might be thought that a factor table should not rank high among mathematical tables for utility, but this measure of comparative value is enormously increased, when we remember the extreme laboriousness with which the individual results would be obtained, when required, in the absence of a table. It would require, for example, a long day's work to prove that 5,999,993 is a prime, or even to resolve into factors 5,999,999, which has no factor less than 1000; in the latter case the verification would be simple enough, but in

the former the whole calculation would have to be duplicated before we could put any confidence in the result. In fact the contrast between the difficulty of obtaining the individual results without a table and the comparative ease with which the table, as a whole, is constructed is most striking.

A portion of Burckhardt's introduction relates to the calculation of the logarithms of large numbers by means of a factor table in connection with such tables as Wolfram's table of the hyperbolic logarithms of primes up to 10,009 to 48 decimal places; and the introduction to the fourth million contains an example, the principle being to multiply by a factor whose logarithm is known, such that the product shall be within the range of the factor table, and shall on trial be found to differ by only one or two units from a composite number, x, all of whose prime factors are within the range of the logarithmic table. The logarithm is then obtained by the use of the formula

$$\log(x+a) = \log x + \frac{a}{x} - \tfrac{1}{2}\frac{a^2}{x^2} + \tfrac{1}{3}\frac{a^3}{x^3} - \text{etc.},$$

in which, of course, the object is to render $\frac{a}{x}$ as small as possible.

"But," to quote Mr. J. W. L. Glaisher (*Camb. Phil. Proc.*, *Feb.* 11, 1878), "the real importance of an extended factor table lies in the fact that such a table is a fundamental one in the Theory of Numbers. The number of factors of a number, their sum, etc., are elements which enter into many results in the Theory of Numbers, and it is clear that, even on this account alone, it is desirable to have a table by means of which the resolution of high numbers into their factors is rendered practicable, so that suitable examples, verifications, etc., of such theorems may be obtainable. Also, conjectural theorems of this nature could not readily be tested without such tables. Again, the law of frequency of prime numbers has been the subject of analytical investigation, and theoretic formulæ giving the average frequency of primes, and the approximate number of primes between any given limits, have been obtained by Legendre, Tchebycheff, Hargreave, Riemann, and others, and it is essential to have the means of comparing the numbers given by these formulæ with those found by actually counting the primes."

Legendre published in 1808 his formula

$$\frac{x}{\log x - 1.08366}$$

for the number of primes inferior to x. The reasoning by which he arrives at the form

$$\frac{x}{A \log x - B}$$

is vague and unsatisfactory, and no account is given of the method by which

the constants were determined. Legendre gives comparisons of the value of this formula with the number of primes counted, up to 400,000, from Vega's *Tabulæ*. Legendre's confidence in this formula was perhaps based upon the fact that, although the constant may have been determined from the value $x = 10{,}000$, the results were satisfactory up to 400,000. He says, "Il est impossible qu'une formule représente plus fidèlement une série de nombres d'une aussi grande étendue et sujette nécessairement à de fréquentes anomalies." Subsequently Legendre extended his comparisons to 1,000,000 by means of Chernac's *Cribrum* and the results were still satisfactory.

In 1849 Gauss refers, in a letter to Encke, to the value of the constant in Legendre's formula. He had, with the assistance of Goldschmidt, made an enumeration of primes from Burckhardt's three millions. He gives comparisons with Legendre's formula, and calculates the values of the constant which must be used in order to bring the formula into agreement with the numbers counted for $x = 500{,}000$, 1,000,000, etc., to 3,000,000. It appears that the constant must be decreased as x increases. It was subsequently proved by Tchebycheff that the constant must be unity when x is infinite. Gauss also gave comparisons with the logarithmic integral

$$\text{li } x = \int_0 \frac{dx}{\log x}$$

Tchebycheff in 1848 arrived independently at the formula

$$\int_2 \frac{dx}{\log x}$$

and Hargreave in the following year represented the number of primes between x and x' by

$$\int_x^{x'} \frac{dx}{\log x}$$

In 1872, Mr. J. W. L. Glaisher, in a paper read before the British Association, gave the result of enumerations made from Burckhardt's and Dase's Tables, and comparisons, throughout the second and ninth millions, with the differences of the values of the integral li x. The enumerations of Gauss were found to be very inaccurate. In 1876 and 1877, in communications to the Cambridge Philosophical Society, Mr. Glaisher, having carefully revised the enumerations, gives the results for the six millions which had then been published, and states his purpose of comparing the results with Legendre's formula and with the logarithmic integral throughout the six millions. These enumerations were extended to the fourth and fifth millions, and the results appeared in the introductions to those volumes; and the introduction to the sixth million contains tables giving the number of primes in each successive group of 1,000 numbers from 0 to 9,000,000. Also in each group of 10,000 numbers, and in groups of 100,-

000, of 250,000, of 500,000, and in each million. The irregularity of the distribution of primes is strikingly shown by the differences which are added in the tables where the groups contain 100,000 numbers or more; "not even in the table where the groups contain 500,000 numbers do the numbers of primes in the successive groups always decrease, there being an increase in one instance."

The results for the complete millions are as follows:

n	Number of primes in the nth million.	Difference.	Number of primes inferior to n millions.
1	78,499		78,499
		−8066	
2	70,433		148,932
		−2548	
3	67,885		216,817
		−1556	
4	66,329		283,146
		− 960	
5	65,369		348,515
		−1033	
6	64,336		412,851
		− 537	
7	63,799		476,650
		− 641	
8	63,158		539,808
		− 398	
9	62,760		602,568

In 1859 Riemann communicated to the Berlin Academy a paper (which appears at page 136 of his collected works) in which he investigates the law of frequency of primes, and finds for the non-periodic part of the expression for the number of primes inferior to x the formula

$$\operatorname{li} x - \tfrac{1}{2} \operatorname{li} x^{\frac{1}{2}} - \tfrac{1}{3} \operatorname{li} x^{\frac{1}{3}} + \tfrac{1}{6} \operatorname{li} x^{\frac{1}{6}} - \tfrac{1}{7} \operatorname{li} x^{\frac{1}{7}} + \text{etc.},$$

in which the terms are of the form $\pm \frac{1}{n} \operatorname{li} x^{\frac{1}{n}}$, where n is the product of primes all different, and the sign is + or −, according as the number of primes is even or odd. To this memoir the late Professor H. J. S. Smith referred in 1876 as containing "the only investigation of the asymptotic frequency of the primes which can be regarded as rigorous."

The introduction to the sixth million contains the complete comparisons, not only with Legendre's formula, and with li x, but also with Riemann's formula at intervals of 50,000 for the whole nine millions. The tables also give the deviations of the formulas from the actual numbers of primes. These deviations show the great superiority of Riemann's formula, which is found to be more accurate than Legendre's even for the smaller values of x, and to give satisfactory results for the whole nine millions. Legendre's formula, on the other hand, commences at 650,000 to give results which are, with a single exception (at 1,200,000) always in excess, the excess amounting to 391 at 8,450,000 and to 282 at 9,000,000. The formula li x gives results largely in excess, even for the smaller values of x, and steadily diverges, but not so rapidly as Legendre's formula; so that it is more accurate than the latter when x exceeds five millions. The mean deviations derived from the table are, for Riemann's formula, − 9, for li x, + 163, and for Legendre's formula, + 171.

To exhibit the deviations more clearly to the eye, Mr. Glaisher has given a diagram in which successive values of the mean of three deviations in the table are taken as ordinates corresponding to the middle values of x as abscissas. These "deviation curves" are, of course, extremely irregular, owing to the anomalous distribution of the primes; but, while the other curves tend steadily upward, Riemann's curve shows no tendency to cumulative error; in fact it cuts the axis 18 times. Had the actual values of the deviations at intervals of 50,000 been constructed as ordinates, the curve would have been still more irregular, but it would have cut or touched the axis 51 times within the nine millions, the last time at 8,500,000; the deviation at 9,000,000 is — 132.

It is possible to determine the number of primes, for particular values of x, without actually constructing the factor table. The number of multiples of p, including p itself, inferior to x is $E\left(\frac{x}{p}\right)$, the greatest integer which does not exceed $\frac{x}{p}$. Taking account of the numbers divisible by both p_1 and p_2, the number divisible by either p_1 or p_2 is $E\left(\frac{x}{p_1}\right) + E\left(\frac{x}{p_2}\right) - E\left(\frac{x}{p_1 p_2}\right)$; and in general the number divisible by one of the primes $p_1, p_2, \ldots p_n$ is

$$\Sigma E\left(\frac{x}{p_1}\right) - \Sigma E\left(\frac{x}{p_1 p_2}\right) + \Sigma E\left(\frac{x}{p_1 p_2 p_3}\right) - \text{etc.}$$

The employment of this formula it will be noticed is equivalent to calculating the number of entries of one of the primes $p_1, \ldots p_n$ that would be made by the "sieve" method, if a factor table were to be constructed to extend to x, but in the case of primes greater than $\sqrt[3]{x}$ which occur as least factors of two-factor numbers only, within the limit of the table, the number of times p would be entered may be determined by a method analogous to the "multiple" method described above, *i. e.*, it is only necessary to count the primes between p and $\frac{x}{p}$. Thus the whole number of entries may be determined, and by subtraction the number of primes. The process is, of course, very laborious, but Meissel in 1870 and 1871 employed it to calculate the number of primes inferior to one million, to ten millions, and to one hundred millions. Meissel's number for one million was exactly confirmed by the actual count from Burckhardt's table, and Mr. Glaisher regards his results for ten millions and one hundred millions, namely, 664,580 and 5,761,461, as entitled to be accepted with confidence. Accordingly the values of the several formulæ were calculated for $x = 10{,}000{,}000$ and for $x = 100{,}000{,}000$. At ten millions, Riemann's formula gives an excess of 87, so that the deviation curve crosses the axis at least once between nine and ten millions; the deviation of li x is 338, and that of Legendre's formula is 560. At one hundred millions Riemann's deviation is only 90, that of li x is

748, and that of Legendre's formula is 6543. The accuracy of Riemann's formula at this point is extraordinary, the error being less than $\frac{1}{60,000}$ of the actual number of primes.

The introduction to the sixth million, which contains many important and interesting results besides those briefly summarized in this review, concludes with a discussion of the mode of calculating the values of li x and of Riemann's formula.

A mathematical table of this magnitude and importance may well be reckoned one of the "long results of time;" and England may be justly proud that such a work has not only been completed by one of her own mathematicians, but furnished with so worthy a crown.

EXERCISES.

1

A HORIZONTAL WIND blows against a hemispherical dome of radius R'. The pressure exerted by the wind on a plane surface normal to its direction is P pounds to the square foot; on a surface oblique to its direction the pressure exerted is normal, but is reduced in the ratio (Poncelet, *Mécanique Industrielle*, 403)

$$1 : 1 + \tfrac{1}{2} \tan^2 i,$$

where i is the angle of incidence. It is required to find the magnitudes and points of application of the horizontal and vertical components of the resultant wind-pressure. [*W. M. Thornton.*]

2

THREE closely connected tanks, T_1, T_2, T_3, contain Q_1 gallons of water, Q_2 gallons of vinegar, Q_3 gallons of brandy, respectively. A flow is set up from T_1 through T_2 to T_3 and back to T_1 at the rate of 1 gallon per second. The liquids are assumed to mix instantaneously, and the lengths of the connecting pipes are neglected. Show how to calculate the amount of water in each tank at the end of T seconds. [*De Volson Wood.*]

3

To find the radius of a sphere, which being let fall into a given conical wine-glass, full of water, will displace the maximum or minimum quantity of water. [*Benjamin Alvord.*]

4

ON one of the bounding radii of a given quadrant, radius r, a semicircle is drawn, radius $\frac{1}{2}r$, the semicircle being within the quadrant. Find the average area of the circle touching the other bounding radius of the quadrant and the arc of the semicircle. [*Artemas Martin.*]

5

THREE LINES are drawn from a point within a square to three corners of the square. Find a side of the square. [*J. M. Greenwood.*]

6

FIND the value of the definite integral

$$\int_{\frac{1}{2}\pi}^{\pi} (1 - \tfrac{1}{2}\cos^2\vartheta + \sin\vartheta \sin 2\vartheta)^{\frac{1}{2}}.$$

[*L. G. Barbour.*]

PUBLICATIONS RECEIVED.

[In an early number a review will be given of the latest works on Algebra. ED.]

A PRACTICAL ARITHMETIC. By G. A. Wentworth and Thomas Hill. Boston: Ginn, Heath, & Co.

ELEMENTS OF ALGEBRA. By G. A. Wentworth. Boston: Ginn, Heath, & Co.

ELEMENTS OF PLANE AND SOLID GEOMETRY. By G. A. Wentworth. Boston: Ginn, Heath, & Co.

PLANE AND SPHERICAL TRIGONOMETRY. By G. A. Wentworth. Boston: Ginn, Heath, & Co.

SURVEYING AND NAVIGATION. By G. A. Wentworth. Boston: Ginn, Heath, & Co.

A COMPLETE ALGEBRA. By A. Schuyler. Cincinnati: Van Antwerp, Bragg, & Co.

ELEMENTS OF GEOMETRY. By A. Schuyler. Cincinnati: Van Antwerp, Bragg, & Co.

FIVE-PLACE LOGARITHMIC AND TRIGONOMETRIC TABLES. Arranged by G. A. Wentworth and G. A. Hill. Boston: Ginn, Heath, & Co.

A PRACTICAL ARITHMETIC. By Edward Olney. New York: Sheldon & Co.

THE COMPLETE ALGEBRA. By Edward Olney. New York: Sheldon & Co.

ELEMENTARY GEOMETRY. New Edition. By Edward Olney. New York: Sheldon & Co.

UNIVERSITY ALGEBRA. By Webster Wells. Boston: Leach, Shewell, & Sanborn.

ELEMENTS OF SURVEYING AND LEVELING. By Charles Davies. Revised by J. Howard Van Amringe. New York: A. S. Barnes & Co.

THE THEORY OF TURBINES. Abstract. By Robert H. Thurston. (Journal of the Franklin Institute.)

ALIGNMENT CURVES ON ANY SURFACE, WITH SPECIAL APPLICATION TO THE ELLIPSOID. Abstract. By C. H. Kummell. (Philosophical Society of Washington. Mathematical Section.)

THE THEORY OF ERRORS PRACTICALLY TESTED BY TARGET-SHOOTING. Abstract. By C. H. Kummell. (Philosophical Society of Washington. Mathematical Section.)

BALLISTICS. By James M. Ingalls. (Course of Artillery.) Fort Monroe: United States Artillery School.

ON CERTAIN POSSIBLE ABBREVIATIONS IN THE COMPUTATION OF THE LONG-PERIOD INEQUALITIES OF THE MOON'S MOTION DUE TO THE DIRECT ACTION OF THE PLANETS. By G. W. Hill. (American Journal of Mathematics.)

COMPOUND DETERMINANTS. By C. A. Van Velzer. (American Journal of Mathematics.)